Inverse Scattering Method for the Nonvanishing Potential
—— Wave Modulation in a Stable Medium ——

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We derive the inverse scattering method which makes it possible to analyse the wave modulation in a stable medium. Introducing a certain transformation, we make clear the analytical properties of Jost functions and scattering data. According to the AKNS's method, Gel'fand-Levitan integral equations are derived systematically.

1. Introduction.

As the most exciting recent advances in a applied mathematics the inverse scattering method has been developed to solve the initial value problem for certain nonlinear partial differential equations which arise naturally in many scientific areas[1-5].

The phenomena of the wave modulation and self-focusing or self-defocusing are well described by the nonlinear Schrödinger equation[6],

$$iu_t + u_{xx} - \kappa \mid u \mid^2 u = 0 \ . \tag{1.1}$$

This equation has been solved by the inverse scattering method for the unstable case ($\kappa < 0$)[2] and the stable case ($\kappa > 0$)[7].

For the stable case it is important that a potential of the associated eigen value problem does not vanish at infinity. From this reason we meet with a difficulty that a Neumann series of the Jost function does not converge for all x without certain modifications of discussions. To see this situation we take the associated eigen value problem of eq. (1.1) for the stable case[7],

$$\mathrm{v}_x = [-i\lambda\sigma_3 + Q(x)]\,\mathrm{v} \tag{1.2}$$

where v is a column vector, λ is an eigen value and

$$Q(x) = \begin{pmatrix} 0, & q^*(x) \\ -q(x), & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}.$$

A potential q(x) follows to the nonvanishing conditions,

$$q(x) \to e^{i\theta} \quad \text{as } x \to -\infty \text{ and } q(x) \to 1 \text{ as } x \to \infty, \tag{1.3}$$

where θ is a real constant. We can set the asymptotic states ($x \to \pm\infty$) of eq.(1.2) to the following matrix forms,

$$\Phi_0(\lambda,x) = \begin{pmatrix} e^{-i\xi x}, & (\lambda-\xi)\, e^{-i\theta} e^{i\xi x} \\ (\lambda-\xi)\, e^{i\theta} e^{-i\xi x}, & e^{i\xi x} \end{pmatrix} \quad (x=-\infty), \tag{1.4a}$$

$$\Psi_0(\lambda,x) = \begin{pmatrix} e^{-i\xi x}, & (\lambda-\xi)\, e^{i\xi x} \\ (\lambda-\xi)\, e^{-i\xi x}, & e^{i\xi x} \end{pmatrix} \quad (x=+\infty), \tag{1.4b}$$

where ξ is a double-valued function of λ,

$$\xi = \sqrt{\lambda^2-1}\ .$$

We can define Jost (matrix) function $\Phi(\lambda,x)$ and $\Psi(\lambda,x)$ satisfying eq. (1.2) and the following asymptotic conditions,

$$\Phi(\lambda,x) \rightarrow \Phi_0(\lambda,x) \qquad \text{as} \quad x \rightarrow -\infty\ , \tag{1.5a}$$

$$\Psi(\lambda,x) \rightarrow \Psi_0(\lambda,x) \qquad \text{as} \quad x \rightarrow +\infty\ . \tag{1.5b}$$

We remark that the forms (1.4) have off-diagonal elements which make it impossible to expand each element of the Jost matrix to a Neumann series, then it becomes difficult to make clear the analytic region of Jost functions as to λ. Zakharov et. al. had used a triangular representation instead of the Neumann series technique[7], but their method does not solve this problem. Recently the authers of this paper had settled this problem for the Zakharov-Shabat eigen value problem by introducing a certain transformation[8].

In this paper our method is applied to eq. (1.2) and the inverse problem are solved according to the AKNS's method[9].

2. Analytical properties of the Jost functions.

To develope the useful Neumann series discussion, we introduce the following transformation for eq. (1.2),

$$v = A(\lambda,\xi,x)\, \tilde{v}\ , \tag{2.1}$$

where the matrix A defined as

$$A(\lambda,\xi,x) = \begin{pmatrix} 1, & p^*(x)(\lambda-\xi) \\ p(x)(\lambda-\xi), & 1 \end{pmatrix}. \tag{2.2}$$

In the following the parameter ξ is often omitted for simplicity.
We choose the smooth function p(x) with the same asymptotic property as q(x),

$$p(x) \rightarrow \begin{cases} e^{+i\theta} (\text{as} \quad x\rightarrow-\infty)\ , \\ 1 \quad (\text{as} \quad x\rightarrow+\infty)\ , \end{cases} \tag{2.3a}$$

and specify as

$$| p(x) | = 1, \quad p(x) = \exp\{i\varphi(x)\}, \tag{2.3b}$$

for the briefness of discussion.

Now we remark the relations,

$$\Phi_0(\lambda,x) = A^{(-)}(\lambda)\ J(\xi,x), \qquad \Psi_0(\lambda.x) = A^{(+)}(\lambda)\ J(\xi,x)\ , \tag{2.4}$$

where

$$A^{(\pm)}(\lambda) = \lim_{x\to\pm\infty} A(\lambda,x)\ , \qquad J(\xi,x) = \begin{pmatrix} e^{-i\xi x}, & 0 \\ 0, & e^{i\xi x} \end{pmatrix} .$$

These relations suggest that there exist transformed Jost functions $\tilde{\Phi}(\lambda,x)$ and $\tilde{\Psi}(\lambda,x)$ which have the asymptotic state $J(\xi,x)$ without nondiagonal components.

From the substitution of eqs. (2.1) and (2.2) into eq. (1.2), we obtain

$$\tilde{v}_x = \begin{pmatrix} -i\{\xi - a(\lambda,x)\}, & b_1(\lambda,x) \\ b_2(\lambda,x), & i\{\xi - a(\lambda,x)\} \end{pmatrix} \tilde{v}\ , \tag{2.5}$$

where

$$a\,(\lambda,x) = \frac{1}{\xi} f(x) + \frac{\lambda-\xi}{\xi} h(x)\ , \tag{2.6a}$$

$$b_1(\lambda,x) = ip^*(x)\{ig(x) + \frac{1}{\xi}h(x) + \frac{\lambda}{\xi}f(x)\}\ , \tag{2.6d}$$

$$b_2(\lambda,x) = ip(x)\{ig(x) + \frac{1}{\xi}h(x) - \frac{\lambda}{\xi}f(x)\}\ , \tag{2.6c}$$

and

$$f(x) = \frac{1}{2}\{p(x)q^*(x) + p(x)q^*(x) - 2\},$$

$$g(x) = -\frac{i}{2}\{p(x)q^*(x) - p^*(x)q(x)\}\ , \quad h(x) = \frac{1}{2}\varphi_x(x)\ .$$

We note that the functions $a(\lambda,x)$, $b_1(\lambda,x)$ and $b_2(\lambda,x)$ vanish as $|x|\to\infty$. The transformed Jost functions $\tilde{\Phi}(\lambda,x)$ and $\tilde{\Psi}(\lambda,x)$ can be defined as the solution of eq. (2.5) under the boundary conditions,

$$\tilde{\Phi}(\lambda,x) \to J(\xi,x) \qquad \text{as} \quad x\to-\infty, \tag{2.7a}$$

$$\tilde{\Psi}(\lambda,x) \to J(\xi,x) \qquad \text{as} \quad x\to+\infty. \tag{2.7b}$$

In the following, we explain the analytical property of the Jost function $\tilde{\Psi}(\lambda,x)$. We can easily find that the Jost function $\tilde{\Psi}(\lambda,x)$ satisfies the integral equation,

$$\tilde{\Psi}_0^{-1}(\lambda,x)\,\tilde{\Psi}(\lambda,x) = I - \int_x^\infty M(\lambda,y)\ \tilde{\Psi}_0^{-1}(\lambda,y)\tilde{\Psi}(\lambda,y)\ dy\ , \tag{2.8}$$

where I is unit matrix and

$$\tilde{\Psi}_0(\lambda,x) = \begin{pmatrix} e^{-i\alpha(\lambda,x)} & 0 \\ 0, & e^{i\alpha(\lambda,x)} \end{pmatrix} , \tag{2.9a}$$

$$\alpha(\lambda,x)=\xi x+\int_x^\infty a(\lambda,y)dy \ , \tag{2.9b}$$

$$M(\lambda,x)=\begin{pmatrix} 0, & b_1(\lambda,x)\,e^{2i\alpha(\lambda,x)}, \\ b_2(\lambda,x)\,e^{-2i\alpha(\lambda,x)}, & 0 \end{pmatrix} . \tag{2.9c}$$

Making a iteration to eq. (2.8), we can get

$$\widetilde{\Psi}_0^{-1}(\lambda,x)\,\widetilde{\Psi}(\lambda,x)=I-\int_x^\infty M(\lambda,y)dy+\int_x^\infty M(\lambda,y)dy\int_y^\infty M(\lambda,z)\,\widetilde{\Psi}_0^{-1}(\lambda,z)\,\widetilde{\Psi}(\lambda,z)dz. \tag{2.10}$$

If we take the diagonal components of eq. (2.10), the integral equations with closed form can be obtained as

$$\widetilde{\psi}_1(\lambda.x)\,e^{i\alpha(\lambda,x)}=1+\int_x^\infty N(x,y;\lambda)\ \widetilde{\psi}_1(\lambda,y)\,e^{i\alpha(\lambda,y)}dy \ , \tag{2.11a}$$

$$\tilde{\tilde{\psi}}_2(\lambda,x)\,e^{-i\alpha(\lambda,x)}=1+\int_x^\infty N(x,y;\lambda)\,\tilde{\tilde{\psi}}_2(\lambda,y)\,e^{-i\alpha(\lambda,y)}dy \ , \tag{2.11b}$$

and from the nondiagonal components of eq. (2.8) we get

$$\tilde{\tilde{\psi}}_1(\lambda,x)\,e^{i\alpha(\lambda,x)}=-\int_x^\infty b_1(\lambda,y)\,e^{2i\alpha(\lambda,y)}\,\tilde{\tilde{\psi}}_2(\lambda,y)\,e^{-i\alpha(\lambda,y)}dy \ , \tag{2.12a}$$

$$\widetilde{\psi}_2(\lambda,x)\,e^{-i\alpha(\lambda,X)}=-\int_x^\infty b_2(\lambda,y)\,e^{-2i\alpha(\lambda,y)}\,\widetilde{\psi}_1(\lambda,y)\,e^{i\alpha(\lambda,y)}dy \ , \tag{2.12b}$$

where $\widetilde{\psi}_{1,2}$ and $\tilde{\tilde{\psi}}_{1,2}$ are the elements of the matrix $\widetilde{\Psi}$ and

$$N(x,y;\lambda)=b_2(\lambda,y)\,e^{-2i\alpha(\lambda,y)}\int_x^y b_1(\lambda,z)\,e^{2i\alpha(\lambda,z)}dz \ , \tag{2.13a}$$

$$\overline{N}(x,y;\lambda)=b_1(\lambda,y)\,e^{2i\alpha(\lambda,y)}\int_x^y b_2(\lambda,z)\,e^{-2i\alpha(\lambda,Z)}dz \ , \tag{2.13b}$$

Now we may use the next estimation for the Neumann series expansion of eqs. (2.11a) and (2.11b).

$$\begin{cases} |\,\widetilde{\psi}_1(\lambda,x)\,e^{i\alpha(\lambda,x)}\,| \leqq \exp\left(\int_x^\infty |\,N(x,y;\lambda)\,|\,dy\right) \ , \\ |\,\tilde{\tilde{\psi}}_2(\lambda,x)\,e^{-i\alpha(\lambda,X)}\,| \leqq \exp\left(\int_x^\infty |\,\overline{N}(x,y;\lambda)\,|\,dy\right) \ . \end{cases} \tag{2.14}$$

After some caluculations we get

$$\begin{aligned} &|\,\widetilde{\psi}_1(\lambda,x)\,e^{i\xi x}\,| \leqq 2e^{A_0(\lambda,x)}\{e^{B_0(\lambda,x)}+2A_0(\lambda,x)\} \quad (\text{for Im. } \xi<0) \ , \\ &|\,\tilde{\tilde{\psi}}_2(\lambda,x)\,e^{-i\xi x}\,| < 2e^{A_0(\lambda,x)}\{e^{B_0(\lambda,x)}+2A_0(\lambda,x)\} \quad (\text{for Im. } \xi>0) \ , \end{aligned} \tag{2.15}$$

where

$$B_0(\lambda,x)=\int_x^\infty |\,b_0(\lambda,y)\,|\,dy, \quad A_0(\lambda,x)=\int_x^\infty |\,a(\lambda,y)\,|\,dy \ ,$$

$$|\,b_0(\lambda,y)\,| \geqq |\,b_{1,2}(\lambda,y)\,| \ .$$

If we assume the following integrable conditions,

$$\int_{-\infty}^{\infty} f(x)dx<\infty, \quad \int_{-\infty}^{\infty} g(x)dx<\infty, \quad \int_{-\infty}^{\infty} h(x)dx<\infty, \tag{2.16}$$

the quantities of the right hand sides of eq. (2.15) are bounded for fixed λ except for $\zeta=0$ (or $\lambda=\pm 1$). If we introduce branch cuts appropriately between $\lambda=\pm 1$, the function $\zeta(\lambda)$ becomes single-valued and each Jost function becomes differentiable as to λ.

After the similar discussions, we finally get the following theorem.

(Theorem). "If the integrable conditions (2.16) hold, the functions $\tilde{\phi}(\lambda,x)\,e^{i\zeta x}$ and $\tilde{\tilde{\psi}}(\lambda,x)\,e^{-i\zeta x}$ are analytic functions of λ in the upper half ζ-plane (Im. $\zeta>0$), and $\tilde{\tilde{\phi}}(\lambda,x)\,e^{-i\zeta x}$ and $\tilde{\psi}(\lambda,x)\,e^{i\zeta x}$ are analytic in the lower half ζ-plane (Im. $\zeta<0$). Furthermore if the functions f(x), g(x) and h(x) are on compact support, all of these functions become analytic everywhere except for $\zeta=0$."

Where the quantities $\tilde{\phi}$, $\tilde{\tilde{\phi}}$, $\tilde{\psi}$ and $\tilde{\tilde{\psi}}$ are the column vectors as

$$\tilde{\Phi}=(\tilde{\phi}, \tilde{\tilde{\phi}})= \begin{pmatrix} \tilde{\phi}_1, & \tilde{\tilde{\phi}}_1 \\ \tilde{\phi}_2, & \tilde{\tilde{\phi}}_2 \end{pmatrix}, \qquad \tilde{\Psi}=(\tilde{\psi}, \tilde{\tilde{\psi}}) = \begin{pmatrix} \tilde{\psi}_1, & \tilde{\tilde{\psi}}_1 \\ \tilde{\psi}_2, & \tilde{\tilde{\psi}}_2 \end{pmatrix}. \tag{2.17}$$

3. Scattering matrix and asymptotic expansion as to ζ.

we can define the scattering matrix $S(\lambda)$ as

$$\Phi(\lambda,x) = \Psi(\lambda,x)\,S(\lambda), \tag{3.1}$$

where

$$S(\lambda) = \begin{pmatrix} a(\lambda), & \bar{b}(\lambda) \\ b(\lambda), & \bar{a}(\lambda) \end{pmatrix}.$$

From the tansformation (2.1), we can also get

$$\tilde{\Phi}(\lambda,x) = \tilde{\Psi}(\lambda,x)\ S(\lambda). \tag{3.2}$$

From the facts $\det\tilde{\Phi} = \det\tilde{\Psi}=1$, we get

$$\det S(\lambda) = a(\lambda)\,\bar{a}(\lambda)-b(\lambda)\,\bar{b}(\lambda) = 1\ . \tag{3.3}$$

Since $S(\lambda) = \tilde{\Psi}^{-1}(\lambda,x)\,\tilde{\Phi}(\lambda,x)$, the diagonal elements $a(\lambda)$ and $\bar{a}(\lambda)$ become as

$$\begin{cases} a(\lambda) = \tilde{\phi}_1(\lambda,x)\ \tilde{\tilde{\psi}}_2(\lambda,x)-\tilde{\phi}_2(\lambda,x)\ \tilde{\tilde{\psi}}_1(\lambda,x)\ , \\ \bar{a}(\lambda) = \tilde{\tilde{\phi}}_1(\lambda,x)\ \tilde{\psi}_1(\lambda,x)\ -\ \tilde{\tilde{\phi}}_1(\lambda,x)\ \tilde{\psi}_2(\lambda,x).. \end{cases} \tag{3.4}$$

From the theorem in the previous section, we can determine the analytical property of the scattering matrix.

(Theorem). "If the relations (2.16) hold, $a(\lambda)$ and $\bar{a}(\lambda)$ are analytic for Im. $\zeta>0$ and Im.$\zeta<0$, respectively. Furthermore if the functions f(x), g(x) and h(x) are on compact support, all the elemcnts of $S(\lambda)$ are analytic everwhere except for $\zeta=0$".

Using the symmetrical property of eq, (1.2), we can find the next relations,

$$\sigma_1 \Phi^*(\lambda^*,x)\, \sigma_1 = \Phi(\lambda,x)\ , \quad \sigma_1 \Psi^*(\lambda^*,x)\, \sigma_1 = \Psi(\lambda,x)\ , \tag{3.5}$$

where

$$\sigma_1 = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} .$$

Substituting eq. (3.5) into eq. (3.1), we get

$$\sigma_1 S^*(\lambda^*)\, \sigma_1 = S(\lambda). \tag{3.6}$$

Then equation (3.3) becomes as

$$\det\, S(\lambda) = a(\lambda)\, a^*(\lambda^*) - b(\lambda)\, b^*(\lambda^*) = 1\ . \tag{3.7}$$

Furthermore the eigen value λ corresponding to the bound state is real because equation (1.2) is self-adjoint. Considering eq.(3.7) and the fact that ξ is pure imaginary, we conclude that the zeros of $a(\lambda)$ lie on a interval $-1<\lambda<1$. We can also show that these zeros of $a(\lambda)$ are simple.

Now if we perform the integration by part to eqs. (2.11) and (2.12), the following expansion as to large ξ can be obtalned for the Jost function $\widetilde{\Psi}(\lambda,x)$.

$$\widetilde{\psi}(\lambda,x)\, e^{i\alpha(\lambda,x)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2i\xi} \begin{pmatrix} \int_x^\infty b_1(\lambda,y)\, b_2(\lambda,y) dy \\ -b_2(\lambda,x) \end{pmatrix} + O\left(\frac{1}{\xi^2}\right) \quad (\mathrm{Im}.\xi<0), \tag{3.8a}$$

$$\widetilde{\widetilde{\psi}}(\lambda,x)\, e^{-i\alpha(\lambda,x)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2i\xi} \begin{pmatrix} -b_1(\lambda,x) \\ \int_x^\infty b_1(\lambda,y)\, b_2(\lambda,y) dy \end{pmatrix} + O\left(\frac{1}{\xi^2}\right) \quad (\mathrm{Im}.\xi>0). \tag{3.8b}$$

Furthermore we obtain

$$\left\{ \begin{aligned} &\widetilde{\phi}(\lambda,x)\, e^{i\beta(\lambda,x)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(\frac{1}{\xi}\right) \qquad (\mathrm{Im}.\xi>0), \\ &\widetilde{\widetilde{\phi}}(\lambda,x)\, e^{-i\beta(\lambda,x)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(\frac{1}{\xi}\right) \qquad (\mathrm{Im}.\xi<0), \end{aligned} \right. \tag{3.9}$$

where

$$\beta(\lambda,x) = \xi x - \int_{-\infty}^{x} a(\lambda,y)\, dy.$$

Substituting eqs. (3.8) and (3.9) into eq. (3.4), we obtain

$$a(\lambda) = \exp\left\{ \int_{-\infty}^{\infty} a(\lambda,y) dy \right\} + O\left(\frac{1}{\xi}\right) \quad (\mathrm{Im}.\xi>0), \tag{3.10a}$$

$$\bar{a}(\lambda) = \exp\left\{ -\int_{-\infty}^{\infty} a(\lambda,y) dy \right\} + O\left(\frac{1}{\xi}\right) \quad (\mathrm{Im}.\xi<0). \tag{3.10b}$$

4. Triangular representation of the Jost function.

In this section, we will verify that the Jost function can be represented by the integral form consisting with the asymptotic solution and λ-independent kernel.

We assume the next forms as to the Jost functions,

$$\Phi(\lambda,x)=\Phi_0(\lambda,x)+\int_{-\infty}^{x} K(x,s)\ \Phi_0(\lambda,s)\,ds\ , \tag{4.1a}$$

$$\Psi(\lambda,x)=\Psi_0(\lambda,x)-\int_{x}^{\infty} L(x,s)\ \Psi_0(\lambda,s)\ ds\ , \tag{4.1b}$$

where the kernel $K(x,y)$ and $L(x,y)$ are independent of λ. Because the case of $\Phi(\lambda,x)$ does not need in the later sections, we only treat the case of $\Psi(\lambda,x)$ in the following. From the fact that $\Psi(\lambda,x)$ satisfy eq. (1.2), we get

$$\begin{aligned}\int_x^{\infty}&\left\{\frac{\partial L(x,s)}{\partial x}+\sigma_3\frac{\partial L(x,s)}{\partial s}\sigma_3+\sigma_3 L(x,s)\sigma_3[i\lambda\sigma_3+D^{(+)}(\lambda)]-iQ(x)L(x,s)\right\}\Psi_0(\lambda,s)\,ds\\ &=\sigma_3 L(x,s)\sigma_3\Psi_0(\lambda,s)\Big|_{s=x}^{s=+\infty}-\left\{\Delta D^{(+)}(x)-L(x,x)\right\}\Psi_0(\lambda.x)\ ,\end{aligned} \tag{4.2}$$

where

$$D^{(+)}(\lambda)=\begin{pmatrix}-i\lambda, i\\ -i,\ i\lambda\end{pmatrix},\qquad \Delta D^{(+)}(x)=\begin{pmatrix}0,\ i[q^*(x)-1]\\ -i[q(x)-1],\ 0\end{pmatrix}.$$

To realize the above relation (4.2), it is sufficient to impose the following relations,

$$\frac{\partial L(x,y)}{\partial x}+\sigma_3\frac{\partial L(x,y)}{\partial y}\sigma_3+\sigma_3 L(x,y)\ \sigma_3[i\lambda\sigma_3+D^{(+)}(\lambda)]-iQ(x)L(x,y)=0, \tag{4.3a}$$

$$\sigma_3 L(x,x)\sigma_3-L(x,x)\ +\Delta D^{(+)}(x)\ =\ 0\ , \tag{4.3b}$$

$$\lim_{y\to\infty}\ \sigma_3 L(x,y)\ \sigma_3\ \Psi_0(\lambda,y)\ =\ 0. \tag{4.3c}$$

If we assume $L(x,\infty)=0$, these relations can be reduced as the following Cauchy problem,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\begin{pmatrix}L_{11}(x,y)\\ L_{22}(x,y)\end{pmatrix}+i\begin{pmatrix}1,-q^*(x)\\ q(x),\ \ 1\end{pmatrix}\begin{pmatrix}L_{12}(x,y)\\ L_{21}(x,y)\end{pmatrix}=0\ , \tag{4.4a}$$

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\begin{pmatrix}L_{12}(x,y)\\ L_{21}(x,y)\end{pmatrix}+i\begin{pmatrix}1,-q^*(x)\\ q(x),\ 1\end{pmatrix}\begin{pmatrix}L_{11}(x,y)\\ L_{22}(x,y)\end{pmatrix}=0, \tag{4.4b}$$

$$\begin{pmatrix}L_{12}(x,x)\\ L_{21}(x,x)\end{pmatrix}=\frac{i}{2}\begin{pmatrix}q^*(x)-1\\ -q(x)+1\end{pmatrix}, \tag{4.4c}$$

where L_{ij} is the element of matrix L. It can be shown that the system (4.4) can be uniquely solved, from which follows the existence of the representations (4.1). Furthermore the system

(4.4) has a symmetry, by virtue of which we obtain

$$\sigma_1 L^*(x,y)\, \sigma_1 = L(x,y)\ . \tag{4.5}$$

5. The inverse scattering problem.

When the potential q(x) has been given, two Jost functions $\Phi(\lambda,x)$ and $\Psi(\lambda,x)$ are uniquely determined, that is, the scattering matrix $S(\lambda)$ also determined from eq. (3.1) (direct scattering process). In this section we construct integral equations (so called Gel'fand-Levitan equation) which make it possible to determine the potential q(x) from the specified scattering data (inverse scattering process).

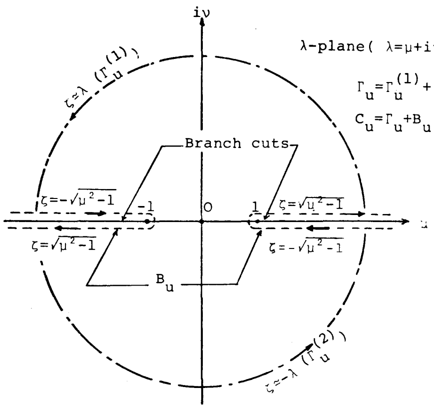
We start from the assumption that f(x), g(x) and h(x) are on compact support, that is, all the components of the Jost functions and scattering matrix are analytical on the whole ζ-plane except for $\zeta=0$.

Now ζ is a double-valued function of λ, $\zeta(\lambda)=\sqrt{\lambda^2-1}$. To make the function $\zeta(\lambda)$ single-valued, we must introduce two Riemann surfaces (called as upper and lower sheets) with a branch cut between two branch points $\lambda=\pm1$. Especially if we choose the branch cuts $(-\infty,-1)$ and $(1,\infty)$ on the λ-plane, the function $\zeta(\lambda)$ maps the upper and lower sheets into the upper half (Im.ζ>0) and lower half (Im.ζ<0) of the ζ-plane, respectively.

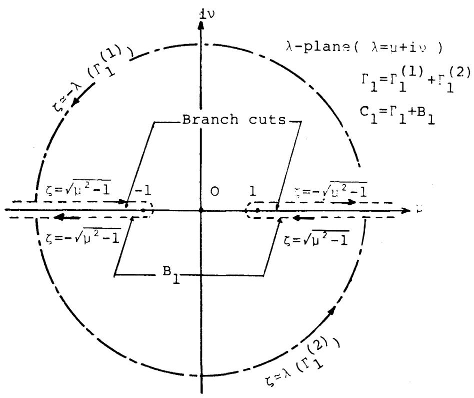
At this stage, we define the contour paths C_u and C_l on the upper sheet and lower sheet of the λ-plane, respectively, as

$$C_u=\Gamma_u+B_u,\quad C_l=\Gamma_l+B_l,$$

$$\Gamma_{u,l}=\Gamma^{(1)}_{u,l}+\Gamma^{(2)}_{u,l},\qquad B_{u,l}=B^{(1)}_{u,l}+B^{(2)}_{u,l}, \tag{5.1}$$



(a) Upper sheet of λ-plane (Im.ζ>0).



(b) Lower sheet of λ-plane (Im.ζ<0)

Fig.1. Two Riemann surfaces and integral paths.

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ (for sufficiently large $|\lambda|$) are defined as counter clock-wise paths on the upper half and lower half of the λ-plane, respectively. On the other hand, $B^{(1)}$ and $B^{(2)}$ (in the neigborhood of the real axis) are defined as clock-wise paths around the cuts $(1,\infty)$ and $(-\infty,-1)$, respectively. These situations are shown schematically in Fig.1.

Now we return to the asymptotic expansions of the Jost functions and scattering data. Substituting the next relations,

$$a(\lambda.x)=\begin{cases} O(\frac{1}{\xi}), & \lambda\epsilon\Gamma_u^{(2)} \text{ or } \Gamma_l^{(2)}, \\ -\varphi_x(x)\ +\ 0\,(\frac{1}{\xi}), & \lambda\epsilon\Gamma_u^{(2)} \text{ or } \Gamma_l^{(1)}, \end{cases} \tag{5.2}$$

into eqs. (3.8), (3.9) and (3.10),, we can explicitly determine the asymptotic forms as shown ir Table I.

Paths \ Functions	Γ_u:Upper sheet(Im.ζ>0)		Paths \ Functions	Γ_l:Lower sheet(Im.ζ<0)	
	$\Gamma_u^{(1)}$ ($\zeta \simeq \lambda$)	$\Gamma_u^{(2)}$ ($\zeta \simeq -\lambda$)		$\Gamma_l^{(1)}$ ($\zeta \simeq -\lambda$)	$\Gamma_l^{(2)}$ ($\zeta \simeq \lambda$)
$\tilde{\phi}(\lambda,x)e^{i\zeta x}$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}e^{i\{\theta-\varphi(x)\}}$	$\tilde{\bar{\phi}}(\lambda,x)e^{-i\zeta x}$	$\begin{pmatrix}0\\1\end{pmatrix}e^{i\{\varphi(x)-\theta\}}$	$\begin{pmatrix}0\\1\end{pmatrix}$
$\tilde{\bar{\psi}}(\lambda,x)e^{-i\zeta x}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}e^{i\varphi(x)}$	$\tilde{\psi}(\lambda,x)e^{i\zeta x}$	$\begin{pmatrix}1\\0\end{pmatrix}e^{-i\varphi(x)}$	$\begin{pmatrix}1\\0\end{pmatrix}$
$a(\lambda)$	1	$e^{i\theta}$	$\bar{a}(\lambda)$	$e^{-i\theta}$	1

Table 1. Asymptotic values of Jost functions and scattering data.

Now we consider the following complex integral[9]

$$I(\tilde{\phi})=\int_{\Gamma_u}\frac{d\lambda'}{a(\lambda')}\cdot\frac{\tilde{\phi}(x',x)}{x'-\lambda}e^{i\xi' x}, \tag{5.3}$$

where $\xi'=\sqrt{\lambda'^2-1}$ and λ is a point on the two Riemann surfaces of the complex λ'-plane. We note that these twe Riemann surfaces are analytically continuated at the branch cuts, then, even if λ is a point on the lower sheet, we can caluculate $I(\tilde{\phi})$ as follows using the asymptotic estimation of Table I.

$$I(\tilde{\phi})=i\pi(1+e^{-i\varphi(x)})\begin{pmatrix}1\\0\end{pmatrix}. \tag{5.4}$$

From eq. (3.2), $I(\tilde{\phi})$ are rewritten as

$$I(\tilde{\phi})=\int_{\Gamma_u}\frac{d\lambda'}{\lambda'-\lambda}\tilde{\psi}(\lambda',x)e_i^{\xi' x}+\int_{\Gamma_u}\frac{d\lambda'}{\lambda'-\lambda}\cdot\frac{b(\lambda')}{a(\lambda')}\tilde{\bar{\psi}}(\lambda',x)\,e^{i\xi' x}. \tag{5.5}$$

Because $\tilde{\psi}(\lambda',x)\,e^{i\xi' x}$is analytical on the both Riemann surfaces from the assumption of compact support, we can get

$$2\pi i\tilde{\psi}(\lambda,x)\,e^{i\zeta x}=\int_{C_u+C_l}\frac{d\lambda'}{\lambda'-\lambda}\tilde{\psi}(\lambda',x)\,e^{i\zeta' x}.$$

Furthermore we note the next relation,

$$\int_{B_u+B_l}\frac{d\lambda'}{\lambda'-\lambda}\tilde{\psi}(\lambda',x)\,e^{i\zeta' x}=0\ .$$

Using the above two relations, we obtain

$$2\pi i\tilde{\psi}(\lambda,x)\,e^{i\zeta x}=i\pi(1+e^{-i\varphi(x)})\begin{pmatrix}1\\0\end{pmatrix}-\int_{\Gamma_u}\frac{d\lambda'}{\lambda'-\lambda}\tilde{\psi}(\lambda',x)\,e^{i\zeta' x}. \tag{5.6}$$

From eqs.(5.4), (5.5) and (5.6) we finally obtain the following integral representation about Jost functions on the complex λ-plane.

$$\tilde{\psi}(\lambda,x)\,e^{i\zeta x}=(1+e^{-i\varphi(x)})\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{2\pi i}\int_{\Gamma_u}\frac{d\lambda'}{\lambda'-\lambda}\cdot\frac{b(\lambda')}{a(\lambda')}\;\tilde{\bar{\psi}}(\lambda',x)\,e^{i\zeta' x}. \tag{5.7}$$

Similarly, considering the integral,

$$I(\tilde{\bar{\phi}})=\int_{\Gamma_l}\frac{d\lambda'}{\bar{a}(\lambda')}\cdot\frac{\tilde{\bar{\phi}}(\lambda',x)}{\lambda'-\lambda}\;e^{-i\zeta' x}$$

we can also obtain

$$\tilde{\bar{\psi}}(\lambda,x)\,e^{-i\zeta x}=(1+e^{i\varphi(x)})\begin{pmatrix}0\\1\end{pmatrix}-\frac{1}{2\pi i}\int_{\Gamma_l}\frac{d\lambda'}{\lambda'-\lambda}\cdot\frac{\bar{b}(\lambda')}{\bar{a}(\lambda')}\;\tilde{\psi}(\lambda',x)\,e^{-i\zeta' x}. \tag{5.8}$$

At this stage, we can derive the Gel'fand-Levitan equation. Operating the transformation $A(\lambda,x)$ to eq.(5.7) and from the triangular repretation (4.1b), we can list up the following three equations,

$$\psi(\lambda,x)\,e^{i\zeta x}=(1+e^{-i\varphi(x)})\,A(\lambda,x)\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{2\pi i}\int_{\Gamma_u}\frac{d\lambda'}{\lambda'-\lambda}\cdot\frac{b(\lambda')}{a(\lambda')}A(\lambda,x)\,A^{-1}(\lambda',x)\,\bar{\psi}(\lambda',x)\,e^{i\zeta' x}, \tag{5.9a}$$

$$\psi(\lambda,x)=A^{(+)}(\lambda)\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta x}-\int_x^\infty L(x,s)A^{(+)}(\lambda)\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta s}\,ds, \tag{5.9b}$$

$$\bar{\psi}(\lambda,x)=A^{(+)}(\lambda)\begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta x}-\int_x^\infty L(x,s)A^{(+)}(\lambda)\begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta s}\,ds. \tag{5.9c}$$

Substituting eqs.(5.9b) and (5.9c) into eq.(5.9a), we get

$$\begin{aligned}A^{(+)}(\lambda)\begin{pmatrix}1\\0\end{pmatrix}-e^{i\zeta x}\int_x^\infty L(x,s)A^{(+)}(\lambda)\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta s}ds&=(1+e^{-i\varphi(x)})A(\lambda,x)\begin{pmatrix}1\\0\end{pmatrix}\\&-\frac{1}{2\pi i}\int_{\Gamma_u}\frac{d\lambda'}{\lambda'-\lambda}\cdot\frac{b(\lambda')}{a(\lambda')}\,A(\lambda,x)A^{-1}(\lambda',x)A^{(+)}(\lambda')\begin{pmatrix}0\\1\end{pmatrix}e^{2i\zeta' x}\\&+\frac{1}{2\pi i}\int_{\Gamma_u}\frac{d\lambda'}{\lambda'-\lambda}\cdot\frac{b(\lambda')}{a(\lambda')}A(\lambda,x)A^{-1}(\lambda',x)\,e^{i\zeta' x}\int_x^\infty L(x,s)A^{(+)}(\lambda')\begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta' s}\,ds.\end{aligned} \tag{5.10}$$

Now we operate the integrator,

$$\frac{1}{4\pi}\int_{B_u}\frac{e^{i(y-x)\zeta}}{\zeta}\,d\lambda\qquad(x<y), \tag{5.11}$$

to the both sides of eq.(5.10). Furthermore using the following relation and difinition,

$$\frac{1}{4\pi i}\int_{B_u}\frac{e^{i(y-x)\zeta}}{\zeta(\lambda'-\lambda)}A(\lambda,x)d\lambda=-\frac{A(\lambda',x)}{\zeta'}\,e^{i(y-x)\zeta'},$$

$$\frac{1}{4\pi}\int_{B_u}\frac{e^{i(y-x)\xi}}{\xi}A^{(+)}(\lambda)\ d\lambda=\sigma(y-x)\begin{pmatrix}0,1\\1,0\end{pmatrix}, \tag{5.12}$$

$$F(x)=\frac{1}{4\pi}\int_{\Gamma_u}\frac{b(\lambda)}{a(\lambda)}\cdot\frac{e^{i\xi z}}{\xi}A^{(+)}(\lambda)d\lambda\ , \tag{5.13}$$

we finally obtain the Gel'fand-Levitan integral equation,

$$L(x,y)\begin{pmatrix}0\\1\end{pmatrix}+F(x+y)\begin{pmatrix}0\\1\end{pmatrix}-\int_x^\infty L(x,s)\,F(y+s)\begin{pmatrix}0\\1\end{pmatrix}ds=0,\ (x<y). \tag{5.14}$$

Repeating a similar process, we can also get

$$L(x,y)\begin{pmatrix}1\\0\end{pmatrix}-G(x+y)\begin{pmatrix}1\\0\end{pmatrix}+\int_x^\infty L(x,s)\,G(y+s)\begin{pmatrix}1\\0\end{pmatrix}ds=0,\ (x<y)\ , \tag{5.15}$$

where

$$G(z)=\frac{1}{4\pi}\int_{\Gamma_l}\frac{\bar{b}(\lambda)}{\bar{a}(\lambda)}\cdot\frac{e^{-i\xi z}}{\xi}\ A^{(+)}(\lambda)d\lambda \tag{5.16}$$

Now we remark that two kinds of spector functions F(z) and G(z) show the following symmetrical property using eq.(3.6),

$$F^*(z)\ =-G(z)\ . \tag{5.17}$$

From this fact, we again get the symmetrical relation (4.5) comparing the two equations (5.14) and (5.15). Clearly the two equations (5.14) and (5.15) are equivalent, then we treat only eqs. (5.13) and (5.14) in the following. .

The spector function F(z) must be reduced to the form which consists of measurable scattering data. We can rewrite this as the form which have discrete spectrum and continuous spectrum, because the integrand of F(z) have only finite simple poles (corresponding to the zeros of $a(\lambda)$) as the singular points.

$$F(z)=\frac{1}{2}\sum_{n=1}^{N}d_n e^{-\eta_n z}A^{(+)}(\mu_n)-\frac{1}{4\pi}\int_{B_u}c(\lambda)\frac{e^{i\xi z}}{\xi}A^{(+)}(\lambda)dx, \tag{5.18}$$

where

$$d_n=\frac{b(\mu_n)}{\eta_n a'(\mu_n)},\quad C(\lambda)=\frac{b(\lambda)}{a(\lambda)}\ ,$$

and where $\mu_n(=\lambda_n)$ is the real zero of $a(\lambda)$ and $\eta_n=\sqrt{1-\mu_n^2}$. If we carry out the integration along the branch cuts, we can get

$$F(z)\begin{pmatrix}0\\1\end{pmatrix}=F_d(z)-F_c(z), \tag{5.19}$$

where

$$F_d(z)=\frac{1}{2}\begin{pmatrix}f_d^{(2)}(z)+if_d^{(1)\prime}(z)\\ f_d^{(1)}(z)\end{pmatrix}, \tag{5.20a}$$

$$f_d^{(1)}(z)=\sum d_n e^{-\eta_n z},\ f_d^{(2)}(z)=\sum\mu_n d_n e^{-\eta_n z},$$

$$F_c(z)=\frac{1}{2}\begin{pmatrix}f_c^{(2)}(z)+if_c^{(1)\prime}(z)\\ f_c^{(1)}(z)\end{pmatrix}, \tag{5.20b}$$

$$f_c^{(1)}(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c(\mu,\xi)-c(-\mu,\xi)}{\mu}e^{i\xi z}d\xi,\quad f_c^{(2)}(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{c(\mu,\xi)+c(-\mu,\xi)\}e^{i\xi z}d\xi.$$

Now we supplement another symmetical property about direct scattering. In this discussions, we must reform the notations about Jost functions and scattering data as follows.

$$\Phi(\lambda,\xi;x)=\Psi(\lambda,\xi;x)\,S(\lambda,\xi)\ , \tag{5.21}$$

and instead of eqs. (1.5a) and (1.5b),

$$\begin{cases} \Phi(\lambda,\xi;x)\to A^{(-)}(\lambda,\xi)\,J(\xi,x) & \text{as } x\to-\infty\ , \\ \Psi(\lambda,\xi;x)\to A^{(+)}(\lambda,\xi)\,J(\xi,x) & \text{as } x\to+\infty\ . \end{cases} \tag{5.22}$$

and where $A(\lambda,\xi;x)$ is defined as same as in eq.(2.2). We can find the symmetrical property as to $A(\lambda,\xi;x)$,

$$A(\lambda,-\xi;x)\,B(\lambda,\xi;x)=A(\lambda,\xi;x)\ , \tag{5.23}$$

where

$$B(\lambda,\xi;x)=\begin{pmatrix} 0, & p^*(x)\,(\lambda-\xi) \\ p(x)\,(\lambda-\xi), & 0 \end{pmatrix}=B^{-1}(\lambda,-\xi;x)\ .$$

We note that the Jost functions $\Phi(\lambda,-\xi;x)$ and $\Psi(\lambda,-\xi;x)$ are also the solutions of eq. (1.2). Then from the considerations of eq. (5.22) the next relations can be obtain,

$$\begin{cases} \Phi(\lambda,\xi;x)=\Phi(\lambda,-\xi;x)\,B^{(-)}(\lambda,\xi), \\ \Psi(\lambda,\xi;x)=\Psi(\lambda,-\xi;x)\,B^{(+)}(\lambda,\xi). \end{cases} \quad (B^{(\pm)}(\lambda,\xi)=\lim_{x\to\pm\infty}B(\lambda,\xi;x).\,) \tag{5.24}$$

Substituting eq. (5.24) into eq. (5.21), we can get the following symmetrical property about the scattering data,

$$S(\lambda,\xi)=B^{(+)}(\lambda,-\xi)\,B^{(-)}(\lambda,\xi). \tag{5.25}$$

From this relations and eq. (3.6), we finally get

$$c(\lambda,\xi)=\frac{b(\lambda,\xi)}{a(\lambda,\xi)}=\frac{b^*(\lambda^*,-\xi^*)}{a^*(\lambda^*,-\xi^*)}=c^*(\lambda^*,-\xi^*). \tag{5.26}$$

When λ and ξ are real, this becomes as $c(\mu,\xi)=c^*(\mu,-\xi)$, where $\lambda=\mu+i\nu$ and $\xi=\xi+i\eta$. After all this remarks that the functions $f_c^{(1)}(z)$ and $f_c^{(2)}(z)$ of eq.(5.20b) are real.

6. Concluding Remarks.

Introducing a transformation (2.2), we made clear the analytical property of eq.(1.2) associated with eq.(1.1). According to AKNS's method we derived the Gel'fand-Levitan integral equation which solves the inverse problem of eq.(1.2).

As far as we used the reduced spector function (5.18), we can remove the assumption that three functions f(x), g(x) and h(x) are on compact support. If we neglect the continuous part of eq.(5.18), the integral equation (5.14) can be solved exactly and the interaction of solitons (envelope solitons) can be made clear[7]. The existence of solitons entirely depends on the contribution from zeros of the diagonal element of scattering matrix. These zeros lie on a real interval between two branch point. Then, if the potential vanishes at infinity, any soliton can not appear.

We can also consider the unstable case of eq.(1.1). In this case the associated eigen value equation is not self-adjoint. If the potetial does not vanish, two branch points appear on the imaginary axsis of the complex Riemann plane. Taking a cut between two branch points, we can also derive the Gel'fand-Levitan equation. The details of this case will be reported elsewhere.

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